Exercises on Chapters 10-12

Lisbon, 2010

1. Let W be a scalar Wiener process under P. Consider a market consisting of two assets with prices S_t^1 and S_t^2 as well as a bank account with constant short rate r. Assume that the model for this market has the form

$$\begin{aligned} dS_t^1 &= \alpha_1 S_t^1 dt + \sigma_1 S_t^1 dW_t \\ dS_t^2 &= \alpha_2 S_t^2 dt + \sigma_2 S_t^2 dW_t \\ dB_t &= rB_t dt \end{aligned}$$

   (a) Discuss informally absence of arbitrage and completeness in this model, using the Meta Theorem.

   (b) Use the martingale approach to provide precise conditions for absence of arbitrage and completeness in the model.

2. Consider a filtered probability space $(\Omega, \mathcal{F}, P, \mathbf{F})$ and, apart from the measure P also a measure $Q \sim P$. Define the likelihood process

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t, \quad 0 \leq t \leq T$$

   Let X be an adapted process. Prove that X is a Q-martingale if and only if LX is a P-martingale.

3. Let D be a stochastic discount factor for a financial market. Prove that the process

$$D_t \Pi_t$$

   is a P-martingale for every price process Π.

4. Consider the following SDE.

$$\begin{aligned} dX_t &= \alpha f(X_t)dt + \sigma(X_t)dW_t, \\ X_0 &= x_0. \end{aligned}$$

   Here f and σ are known functions, whereas α is an unknown parameter. We assume that the SDE possesses a unique solution for every fixed choice of α.

   Construct a dynamical statistical model for this problem and compute the ML estimator process $\hat{\alpha}_t$ for α, based upon observations of X.

5. Consider a financial market consisting of an arbitrary number of assets as well as a bank account constant short rate r. Now consider a fixed price process S_t with P-dynamics

$$dS = \alpha_t S_t dt + S_t \sigma_t dW_t.$$

We furthermore consider a T-claim Y of the form

$$Y = S_T \cdot X$$

where X is a random variable in $\mathcal{F}_T$.

The standard risk neutral valuation formula looks like

$$\Pi_0 [Y] = E^Q \left[e^{-rT} S_T \cdot X \right]$$

(a) Prove that

$$E^Q \left[\frac{e^{-rT} S_T}{S_0} \right] = 1$$

(b) Define the random variable L_T by

$$L_T = \frac{e^{-rT} S_T}{S_0}$$

Since $L_T > 0$ and $E^Q [L_T] = 1$ we may use L_T as a Radon-Nikodym derivative and define a new measure Q^S by

$$dQ^S = L_T dQ, \quad \text{on } \mathcal{F}_T$$

Write down a pricing formula for $\Pi_0 [Y]$ using the new measure Q^S.

(c) Define the process L_t by

$$L_t = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t, \quad 0 \leq t \leq T$$

Derive a formula for L_t and prove that L has Q-dynamics

$$dL_t = L_t \sigma_t dW_t^Q$$

where W^Q is Q-Wiener.

Hint: Use the fact that L is a Q-martingale (why?).

(d) Write down a pricing formula for $\Pi_t [Y]$ using the Q^S measure.

(e) Let A_t be a price process for a non dividend paying asset. Prove that the process

$$\frac{A_t}{S_t}$$

is a Q^S martingale.

Mathematical Finance

November 24, 2012

All notation must be clear. Arguments should be complete.

1. Consider a standard Black-Scholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt \end{aligned}$$

and a simple T-claim X of the form

$$X = \Phi(S_T)$$

Show that it is possible to replicate the claim by a portfolio based on B and S. Also derive an expression for the replicating portfolio.

2. Consider a financial market consisting of a non dividend paying stock with price process S_t and a bank account B_t. We consider a fixed risk neutral martingale measure Q and use the notation

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t.$$

Let Q^S denote the martingale measure for the numeraire S and let L^Sdenote the likelihood process

$$L_t^S = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t.$$

Your task is to derive an expression for the process L_t^S.

3. Let the stock prices S_1 and S_2 be given as the solutions to the following system of SDE:s.

$$\begin{aligned} dS_1 &= \alpha S_1 dt + \delta S_1 dW, \quad S_1(0) = s_1, \\ dS_2 &= \beta S_2 dt + \gamma S_2 dV, \quad S_2(0) = s_2, \end{aligned}$$

The Wiener processes W and V are assumed to be independent. The parameters α, δ, γ, β are assumed to be known and constant. Your task is to price a **minimum option**. This T-claim is defined by

$$X = \min\left[S_1(T), S_2(T)\right]$$

The pricing function for a European call option in the Black-Scholes model is assumed to be known, and is denoted by $c(s, t; K, \sigma, r, T)$ where σ is the volatility, K is the strike price and r is the short rate. You are allowed to express your answer in terms of this function, with properly derived values for K, σ and r.

4. Consider the problem to maximize

$$E\left[\int_0^T f(X_s, u_s)ds + F(X_T)\right]$$

given the controlled diffusion

$$dX_t = \mu(X_t, u_t)dt + \sigma(X_t, u_t)dW_t$$

The state X and the control u are scalar and there are no constraints on u apart from the fact that we restrict ourselves to feedback controls of the form $u(t, X_t)$.

Derive the Hamilton-Jacobi-Bellman equation for this problem.

Mathematical Finance

November 15, 2011

All notation must be clear. Arguments should be complete.

1. Consider a standard Black-Scholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt \end{aligned}$$

   and a simple T-claim X of the form

$$X = \Phi(S_T)$$

   derive the Black-Scholes pricing PDE, including boundary condition, for the claim.

2. Consider again standard Black-Sholes model as above. Your task is to compute the price process $\Pi(t; X)$ of an **asset-or-noting option**. This T-claim is defined by

$$X = S_T \cdot I\{S_T \geq K\},$$

   where I denotes the indicator. In words: If the stock price ends up above the strike K you get the value of the stock, otherwise you get nothing.

3. Consider a model for two countries. We then have a domestic market (Austria) and a foreign market (Japan). The domestic and foreign interest rates, r_d and r_f, are assumed to be given real numbers. Consequently, the domestic and foreign savings accounts satisfy

$$B_t^d = e^{r_d t}, \qquad B_t^f = e^{r_f t},$$

   where B^d and B^f are denominated in units of domestic and foreign currency, respectively. The exchange rate process X, which is used to convert foreign payoffs into domestic currency (the "Euro/Yen"-rate), is modelled by the following stochastic differential equation under the objective measure P

$$dX_t = \mu X_t dt + \sigma X_t d\bar{W}_t,$$

where μ and σ are assumed to be constants and $\bar{W}$ is a P-Wiener process.

A *domestic martingale measure*, Q^d, is a measure which is equivalent to the objective measure P and which makes all a priori given price process, expressed in units of domestic currency and discounted using the domestic risk-free rate, martingales. We assume that if you buy the foreign currency this is immediately invested in a foreign bank account. All markets are assumed to be frictionless.

(a) Derive the Q^d-dynamics of X.

(b) Now take the viewpoint of a foreign-based investor, that is an investor who consistently denominates her profits and losses in units of foreign currency. A *foreign martingale measure*, Q^f, is a measure which is equivalent to the objective measure P and which makes all a priori given price process, expressed in units of foreign currency and discounted using the foreign risk-free rate, martingales.
Find the Girsanov transformation between Q^d and Q^f.

(c) The domestic (foreign) market is said to be *risk neutral* if the domestic (foreign) martingale measure is equal to the objective measure P. Under which conditions are both markets risk neutral?

4. Consider the problem to maximize

$$E\left[\int_0^T f(X_s, u_s)ds + F(X_T)\right]$$

given the controlled diffusion

$$dX_t = \mu(X_t, u_t)dt + \sigma(X_t, u_t)dW_t$$

(a) Write down the HJB equation for this problem (you do not have to derive it).

(b) Define, for each feedback control law $\mathbf{u}$ the process $C_t^{\mathbf{u}}$ by

$$C_t^{\mathbf{u}} = \int_0^t f(X_s^{\mathbf{u}}, \mathbf{u}_s)ds + V(t, X_t^{\mathbf{u}}).$$

where V is the optimal value function for the stochastic control problem, $X^{\mathbf{u}}$ is the process generated by the control $\mathbf{u}$, and $\mathbf{u}_s$ is

Stochastic Finance
in Discrete and Continuous Time

Master in Mathematical Finance

November 22, 2010

Reply each question is a separate sheet. Identify each sheet with your name and number.

You must reply in English to the questions of the Continuous Time Finance section, but you are allowed to reply in Portuguese to the Discrete Time Finance section.

All notation must be clear. Arguments should be complete.

A. Continuous Time Finance (2h)

1. Consider a standard Black-Sholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt \end{aligned}$$

where W is P-Wiener. The short rate r is assumed to be constant. Derive the Black-Scholes PDE for the pricing of a T-claim of the form $\Phi(S_T)$.

2. Consider a financial market consisting of a non dividend paying stock with price process S_t and a bank account B_t. We consider a fixed risk neutral martingale measure Q and use the notation

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t.$$

Let Q^S denote the martingale measure for the numeraire S and let L^Sdenote the likelihood process

$$L_t = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t.$$

Your task is to derive an expression for the process L_t^S.

3. Consider a standard Black Scholes model of the form

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt \end{aligned}$$

(a) Can the process Y, defined (for all $t \geq 0$) by,

$$Y_t = \frac{1}{S_t},$$

be the arbitrage free price of a traded derivative (without dividends) in this model? You need to give an answer and a good argument.

(b) Consider a fixed exercise date T, and define the contingent T-claim X by

$$X = \frac{1}{S_T}.$$

Derive an expression for the arbitrage free price process $\Pi_t[X]$ for this claim.

4. Consider the problem to maximize

$$E\left[\int_0^T f(X_s, u_s)ds + F(X_T)\right]$$

given the controlled diffusion

$$dX_t = \mu(X_t, u_t)dt + \sigma(X_t, u_t)dW_t$$

(a) Write down the HJB equation for this problem (you do not have to derive it).

(b) Define, for each feedback control law $\mathbf{u}$ the process $C_t^{\mathbf{u}}$ by

$$C_t^{\mathbf{u}} = \int_0^t f(X_s^{\mathbf{u}}, \mathbf{u}_s)ds + V(t, X_t^{\mathbf{u}}).$$

where V is the optimal value function for the stochastic control problem, $X^{\mathbf{u}}$ is the process generated by the control $\mathbf{u}$, and $\mathbf{u}_s$ is shorthand for $\mathbf{u}(s, X_s^{\mathbf{u}})$. Prove that $C^{\mathbf{u}}$ is a supermartingale for every control law $\mathbf{u}$, and that $C^{\hat{\mathbf{u}}}$ is a martingale, where $\hat{\mathbf{u}}$ is the optimal control.

Hint: If a process Y has a stochastic differential of the form

$$dY_t = \alpha_t dt + \beta_t dW_t,$$

then Y is a supermartingale if and only if $\alpha_t \leq 0$ for all t with probability one. You may use this result without proof.

B. Discrete Time Finance (0.5h)

1. (a) Consider a derivative security that permits its owner to sell one share of stock for payment K at any time up to and including N, but if the owner does not sell by time N then she must do so at time N. Show that the optimal exercise policy is to sell the stock at time zero and that the value of this derivative security is $K - S_0$. (Assume $r \geq 0$.)

   (b) Explain why a portfolio that holds the derivative security in (a) and a European call with strike K and expiration N is at least as value as an American put struck at K with expiration time N. Denote the time-zero value of the European call by V_0^{EC} and the time-zero value of the American put by V_0^{AP}. Conclude that the upper bound $V_0^{AP} \leq K - S_0 + V_0^{EC}$ on V_0^{AP} holds.

2. Consider a stopping time τ. Show that in discrete time you can replace $\{\tau \leq t\} \in \mathcal{F}_t$, for $all\, t \geq 0$ by $\{\tau = t\} \in \mathcal{F}_t$, for $all\, t \geq 0$.

3. Consider the following problem: $\max_{0 \leq \tau \leq T} E[Z_\tau]$. The optimal value process V is the solution of the following backward recursion

$$\begin{aligned} V_n &= \max\{Z_n, E[V_{n+1}|\mathcal{F}_n]\} \\ V_T &= Z_T. \end{aligned}$$

   Prove that the optimal value process V is the snell envelope of the payout process Z.

Stochastic Finance

December 15, 2009

Reply each question is a separate sheet. Identify each sheet with your name and number. You must reply in English to the questions of the Continuous Time Finance section.

All notation must be clear. Arguments should be complete.

A. Continuous Time Finance (2h)

1. Consider the following boundary value problem in the domain $[0,T]\times R$ for an unknown function $F(t,x)$.

$$\begin{aligned}\frac{\partial F}{\partial t}(t,x)+\mu(t,x)\frac{\partial F}{\partial x}(t,x)+\frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x)+k(t,x) &= 0,\\ F(T,x) &= \Phi(x).\end{aligned}$$

Here μ, σ, k and Φ are assumed to be known functions.

(a) Derive a Feynman-Kač representation for this problem. In this formula it must be quite clear exactly at which points the various functions should be evaluated.[1] .... [2v]

(b) Solve explicitly the following concrete problem

$$\begin{aligned}\frac{\partial F}{\partial t}+\mu x\frac{\partial F}{\partial x}+\frac{1}{2}\sigma^2 x^2\frac{\partial^2 F}{\partial x^2} &= 0,\\ F(T,x) &= \ln(x^2).\end{aligned}$$

.... [2v]

2. Let the stock prices S_1 and S_2 be given as the solutions to the following system of SDE:s.

$$\begin{aligned} dS_1 &= \alpha S_1 dt+\delta S_1 dW, \quad S_1(0)=s_1,\\ dS_2 &= \beta S_2 dt+\gamma S_2 dV, \quad S_2(0)=s_2,\end{aligned}$$

The Wiener processes W and V are assumed to be independent. The parameters α, δ, γ, β are assumed to be known and constant.

[1] If you think this problem is hard you may (with loss of 1v) assume that $k(t,x)=0$.

(a) What are the dynamics of both S_1 and S_2 under the risk neutral measure Q? .... [0.5v]

(b) What are the dynamics of $Z_1 = \frac{S_1}{S_2}$ under the Q^{S_2} measure? Q^{S_2} here denotes the EMM where the price process S_2 is the *numeraire*. .... [0.5v]

(c) Derive an expression for the likelihood process

$$L_t = \frac{dQ^{S_2}}{dQ}, \quad \text{on } \mathcal{F}_t ,$$

with Q^{S_2} as in (b). .... [2v]

(d) Your task now is to price a **minimum option**. This T-claim is defined by

$$X = \min\left[S_1(T), S_2(T)\right]$$

The pricing function for a European call option in the Black-Scholes model is assumed to be known. That is, you are allowed to express your answer in terms of this function, with properly derived values for K, σ and r. .... [3v]

3. (a) Consider the problem to maximize

$$E\left[\int_0^T f(X_s, u_s)ds + F(X_T)\right]$$

given the controlled diffusion

$$dX_t = \mu(X_t, u_t)dt + \sigma(X_t, u_t)dW_t$$

The state X and the control u are scalar and there are no constraints on u apart from the fact that we restrict ourselves to feedback controls of the form $u(t, X_t)$. **Derive** the Hamilton-Jacobi-Bellman equation for this problem. .... [3v]

(b) Consider now a standard Black-Sholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

and denote by X the value of a self-financing portfolio.

Find the optimal investment profile which maximizes the log utility

$$E\left[\ln X_T\right] .$$

Interpret and intuitively explain the result. .... [2v]

B. Discrete Time Finance (0.5h)

1. Consider the time interval $[0, T]$ which is divided into n sub-periods of time of equal length. In this framework consider the binomial model where j is the number of upwards and $n-j$ the number of downwards. In order to get the Black-Scholes model from the binomial model, it is assumed that $U = e^{\sigma\sqrt{t/n}}$, $D = e^{-\sigma\sqrt{t/n}}$ and $q = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{t}{n}}$. When $n \to \infty$, what is the limit of the instantaneous rate of return of the stock? Please interpret the assumptions on U and D as well as your result on the limit. .... [1.5v]

2. Consider a two period binomial model where the initial price of the stock is $S_0 = 100$. In each period the price can increase 30%, 20% or decrease 5%. There is a risk-free asset that pays 10%. In this market there exists an European put option and an American put option, both with a maturity of two periods and exercise price 100. .... [2v]

   (a) Establish the range of variation for the value of the European call option (at $t = 0$) that is compatible with the absence of arbitrage opportunities.

   (b) What can you say about the optimal exercise policy of the American put option at $t = 1$? Suppose now that the American put option was not exercised at time $t = 1$. Please establish the range of variation for its value (at $t = 0$) that is compatible with the absence of arbitrage opportunities.

3. Consider a binomial model satisfying the no-arbitrage conditions. There exists an American derivative that can be exercised until time N. If is is exercised at time n ($n = 0, 1, ..., N$) it pays G_n. The price process V_n for this contract by the risk-neutral pricing formula is given by

$$V_n = \max_{\tau \in S_n} \left[\frac{1}{(1+r)^{\tau-n}} G_\tau I_{\tau < N} \right], \quad n = 0, 1, ..., N,$$

   where S_n denotes the set of all stopping times τ that takes values in the set $\{n, n+1, ..., N, \infty\}$. Show that the discounted process $\frac{1}{(1+r)^n} V_n$ is a supermartingale. .... [1.5v]

Black-Scholes formula:

$$F(t,s) = sN\left[d_1(t,s)\right] - e^{-r(T-t)}KN\left[d_2(t,s)\right].$$

Here N is the cumulative distribution function for the $N[0,1]$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

Stochastic Finance

in Discrete and Continuous Time

Tomas Björk

27 June 2008

Answer each question separately. Arguments should be clear and complete. Notation should be clearly defined.

I Discrete Time Stochastic Finance

Exercise 1 Consider a multiperiod binomial model.

a. Show that this model is arbitrage free if and only if $D \leq 1 + r \leq U$.

b. Show that the set of risk neutral probability measures (denoted Q) is a singleton and is given by $q_U = \frac{1+r-D}{U-D}$ and $q_D = \frac{U-(1+r)}{U-D}$

c. Consider now a two period binomial model where the initial price of the stock is $S_0 = 100$, $U = 1.2$, $D = 0.9$ and $r = 10\%$.

   (i) Find the arbitrage free value (at $t = 0$ and $t = 1$) of a European put option with exercise price 110 and maturity two periods.

   (ii) Consider now an American put option with the same exercise price and maturity of the European one. Please find the arbitrage free value of the American put at $t = 0$ and $t = 1$. Is this process a Q martingale? Which kind of process is it?

II Continuous Time Stochastic Finance

Exercise 3 11 June 2008

Exercise 2 Consider a standard Black-Scholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt \end{aligned}$$

Derive the Black-Scholes PDE for the pricing function $F(t, s)$ for a contingent claim X of the form $X = \Phi(S_T)$.

Exercise 5. i) 2006

Exercise 3 Derive a formula for the futures price process $F(t,T)$ for a contingent claim X to be delivered at T.

Exercise 4 Consider a two-country economy (domestic and foreign) with exchange rate X quoted as

$$X = \frac{\text{domestic}}{\text{foreign}}$$

We assume that X has P-dynamics given by

$$dX_t = \alpha_t X_t dt + \sigma_t X_t dW_t,$$

where W is a d-dimensional P-Wiener process. Here α and σ are adapted processes and σ is d-dimensional row vector process. The filtration is the one generated by the Wiener process W. We assume that the domestic and the foreign short rates r^d and r^f are constant and deterministic, and the bank accounts are denoted by B^f and B^d.

Denote the foreign and the domestic martingale measures (with the respective bank accounts as numeraires) by Q^d and Q^f respectively.

Now define the following likelihood processes

$$\begin{aligned} L_t^f &= \frac{dQ^f}{dP}, \quad \text{on } \mathcal{F}_t \\ L_t^d &= \frac{dQ^d}{dP}, \quad \text{on } \mathcal{F}_t \\ L_t &= \frac{dQ^f}{dQ^d}, \quad \text{on } \mathcal{F}_t \end{aligned}$$

a. Derive an expression for L_t. $\varphi^f \to \varphi^d$

b. Derive the Q^d dynanmics for X. $P \to Q^d$

c. Assume that L^d and L^f have P-dynamics of the form

$$\begin{aligned} L_t^d &= L_t^d \left(\varphi_t^d\right)^* dW_t \\ L_t^f &= L_t^f \left(\varphi_t^f\right)^* dW_t \end{aligned}$$

where $*$ denotes transpose. Derive a relation between σ, φ_t^d and φ_t^f.

III Optimal Stopping

Exercise 5 John wants to sell his house. We denote by X_t the price at time t of a John's house and assume it varies according to a stochastic differential equation:

$$dX_t = rX_t dt + \sigma X_t dW_t$$

W is a one-dimensional Wiener process and r and σ are known constants.Let C be the flat tax connected with the sale of the house. Thus, John is trying to maximize the net obtained form the sale of the house

$$e^{-\rho t}(X_t - C)$$

where ρ is the discounting factor. We assume John is a patient man and $\rho < r$. Solve the optimal stopping problem

$$\max_{\tau} E^Q \left[e^{-\rho\tau}(X_\tau - C)\right]$$

and explain when should John sell his house.

....

Black-Scholes formula:

$$F(t,s) = sN\left[d_1(t,s)\right] - e^{-r(T-t)}KN\left[d_2(t,s)\right].$$

Here N is the cumulative distribution function for the $N[0,1]$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

....

Good Luck!

Continuous Time Finance

Test examples
Lisbon 2008

1. Assume that the stock price is assumed to follow standard GBM

$$dS_t = \alpha S_t dt + \sigma S_t dW_t,$$

and the short rate r is assumed to be constant. Consider a derivative security which at time T pays the amount $\frac{1}{S_T}$. Compute the arbitrage free price process $\Pi(t; X)$.

2. Recall that a forward contract on X contracted at time t, with time of delivery T, and with the forward price $f(t; T, X)$ can be seen as a contingent T-claim X with payoff

$$Z = X - f(t; T, X),$$

where the forward price is determined at time t in such a way that the price of Z is zero at time t, i.e. $\Pi(t; Z) = 0$.

   (a) Derive a general formula for the forward price $f(t; T, X)$.

   (b) Now assume that a stock price S is given by the Black-Scholes model and let X be givwen by $X = S_T$. Compute the forward price $f(t; T, S_T)$ assuming the Black-Scholes model for S.

   (c) A *break forward contract* is a T-claim designed to limit the potential loss of a long position in a forward contract by a prespecified amount. More specifically the payoff X from a long position in a break forward is defined by

$$X = \max\{S_T, f(0; T, S_T)\} - K,$$

   where $f(0; T, S_T)$ is the forward price of the stock for settlement at time T, and $K > f(0; T, S_T)$ is some constant. The delivery price K is set in such a way that the break forward contract is worthless when it is entered into.
   Compute the delivery price K.

3. Specify the assumptions in the Black-Scholes model, and derive the Black-Scholes partial differential equation for the price of a simple claim of the form $X = \Phi(S_T)$.

4. Consider the following model for two stock prices (without dividends)

$$\begin{aligned} dS_1 &= \alpha_1 S_1 dt + S_1 \sigma_1 dW, \\ dS_2 &= \alpha_2 S_2 dt + S_2 \sigma_2 dW. \end{aligned}$$

Here α_1 and α_2 are known real numbers, whereas σ_1 and σ_2 are two-dimensional constant row vectors. W is a standard two dimensional Wiener process. The short rate is assumed to be constant. The models is assumed to be arbitrage free. Consider the T-claim X defined by the following points:

- X is specified in terms of the underlying stocks , a strike price K, and an exercise date T.
- The holder of the claim will obtain nothing if $S_2(T) \leq K$.
- If $S_2(T) > K$ then the holder of the claim will obtain $S_2(T) - K$ shares (i.e. units) of asset No 1.

Compute the price $\Pi(t; X)$.

Stochastic Finance

in Discrete and Continuous Time

Tomas Björk

11 June 2008

Answer each question separately. Arguments should be clear and complete. Notation should be clearly defined.

I Discrete Time Stochastic Finance

Exercise 1 Consider a one period model with three assets: a stock, a call option and a riskless asset. At moment $t = 0$ the price of the stock is 100 and at moment $t = 1$ the stock can take three possible values: 100, 130 and 140. The call option has an exercise price of 110 and maturity of one period.

a. Establish the range of variation for the risk-free rate that is compatible with the absence of arbitrage opportunities.

b. Consider that the risk-free rate is 20%. Find the set of risk neutral probability measures (denoted Q) and establish the arbitrage free range of variation for the value of the call option.

II Continuous Time Stochastic Finance

Exercise 2 November 2009

Exercise 2 Consider a standard Black-Scholes model for the stock

$$dS_t = \alpha S_t dt + \sigma S_t d\bar{W}_t,$$

where $\bar{W}$ is P-Wiener. The short rate r is assumed to be constant.

Your task is to compute the price process $\Pi(t; X)$ of an *asset-or-noting option.* This T-claim is defined by

$$X = S_T \cdot I\{S_T \geq K\},$$

where I denotes the indicator. In words: If the stock price ends up above the strike K you get the value of the stock, otherwise you get nothing.
Hint: Change of numeraire is one possibility.

Exercise 3 Consider a standard Black-Scholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt \end{aligned}$$

Prove that every T-claim X claim of the form $X = \Phi(S_T)$ can be replicated.

Exercise 4 Consider a non dividend paying asset with P-dynamics given by

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t,$$

where W is a d-dimensional P-Wiener process. Here α and σ are adapted processes and σ is d-dimensional row vector process. The filtration is the one generated by the Wiener process W. Consider a risk neutral martingale measure Q with the bank account B as the numeraire. Now we want to change numeraire from B to S.

a. Derive an expression for the likelihood process

$$L_t = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t$$

where Q^S is the EMM for the asset S.

b. Suppose furthermore that there is a process X with Q-dynamics

$$dX_t = \mu_t dt + \gamma_t dW_t^Q,$$

where W^Q is Q-Wiener. What will the Q^S dynamics for X look like?

III Optimal Stopping

Exercise 5

a. Define an optimal stopping time τ. Prove that, in discrete time, you can replace the standard condition with

$$\{\tau = n\} \in \mathcal{F}_n.$$

This does not happen in continuous time. Why?

b. We bought an American call written on a foreign stock S^f, struck in domestic currency. The Q-dynamics of the foreign stock are given by

$$dS_t^f = r_f S_t^f dt + \sigma S_t^f dW_t^f$$

and the exchange rate dynamics are given by:

$$dX_t = X_t(r_d - r_f)dt + X_t \sigma_X dW_t$$

Hence, the price of the foreign American call struck in domestic currency is given by:

$$\max_{\tau} E^Q[e^{-r_d \tau} \max\{X_\tau S_\tau - K, 0\}]$$

What can you say about the price of the call? Explain.

....

Black-Scholes formula:

$$F(t,s) = sN\left[d_1(t,s)\right] - e^{-r(T-t)}KN\left[d_2(t,s)\right].$$

Here N is the cumulative distribution function for the $N[0,1]$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

....

Good Luck!

Stochastic Finance

November 10, 2009

Reply each question is a separate sheet. Identify each sheet with your name and number. You must reply in English to the questions of the Continuous Time Finance section.

All notation must be clear. Arguments should be complete.

A. Continuous Time Finance (2h)

✓ **1. Solve the boundary value problem**

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}(t,x) &= 0, \\ F(T,x) &= x^2 - x. \end{aligned}$$

Here σ is a known constant. You may use the Feynman-Kač Representation Theorem without proving it.............................[3v]

2. Consider a standard Black-sholes model for the stock

$$dS_t = \alpha S_t dt + \sigma S_t d\bar{W}_t,$$

where $\bar{W}$ is P-Wiener. The short rate r is assumed to be constant. Your task is to compute the price process $\Pi(t;X)$ of an asset-or-noting option. This T-claim is defined by

$$X = S_T \cdot I\{S_T \geq K\},$$

where I denotes the indicator.
In words: If the stock price ends up above the strike K you get the value of the stock, otherwise you get nothing.....................[4v]

✓ **3. Let W be a standard Wiener process on $(\Omega, \mathcal{F}, \mathbf{F}, P_0)$ where the filtration is the one generated by W. Fix a time interval $[0,T]$. Under the measure P_0, the process X has the dynamics**

$$dX_t = \sigma\sqrt{X_t}dW_t,$$

where σ is a known constant.

Ex: 2 $\Pi(t;X) = S_t E_t^{Q^S}\left(\frac{X}{S_T}\right) = S_t E_t^{Q^S}\left[\frac{S_T \cdot \mathbb{I}\{S_T \geq K\}}{S_T}\right] = S_t E_t^{Q^S}\left[\mathbb{I}\{S_T \geq K\}\right] =$

$= S_t Q^S(S_T \geq K) = S_t Q^S(\ln S_T \geq \ln K) = S_t\left(1 - \Phi\left(\frac{\ln K - \mu}{\beta}\right)\right)$

Em Q: $dS = rSdt + \sigma S dW^Q$, $\varphi = \sigma$

$dS = rSdt + \sigma S(\varphi dt + dW^{Q^S})$

$= (r+\sigma^2)Sdt + \sigma S dW^{Q^S}$

$S_T = S_t e^{(r+\sigma^2 - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}$

$\ln S_T = \ln S_t + (r+\sigma^2 - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)$ $\quad \ln S_T \sim N(\underbrace{\ln S_t + (r+\sigma^2 - \frac{1}{2}\sigma^2)(T-t)}_{\mu}, \underbrace{\sigma\sqrt{T-t}}_{\beta})$

(a) Define, for each real number α, a Girsanov transformation such that the measure P_0 is transformed into a measure P_α, such that X under P_α solves the equation

$$dX = \alpha X dt + \sigma\sqrt{X} dW^\alpha,$$

where W^α is a P_α-Wiener process. Define the corresponding likelihood process L_t^α, where

$$L_t^\alpha = \frac{dP_\alpha}{dP_0}, \quad \text{on } \mathcal{F}_t.$$

Your task is to give a precise description of this measure transformation by specifying the dynamics of L_t^α. .... [2v]

(b) Determine, for every $t \leq T$, the maximum likelihood estimator $\hat{\alpha}(t)$ for the parameter α, based on observations of X over the interval $[0, t]$, i.e. the value of α that maximizes L_t^α. Note that the answer shall be expressed in terms of the process X, and simplified as far as possible. .... [2v]

4. Consider the problem of maximizing logarithmic utility

$$E\left[\int_0^T e^{-\delta t}\ln(c_t)dt + K.\ln(X_T)\right]$$

given the usual wealth dynamics

$$dX_t = \left(u_t^0 r + u_t^1 \alpha\right) X_t dt - c_t dt + u_t^1 \sigma X_t dW_t$$

with r, α, σ constant and the usual control constraints

$$\begin{aligned} c_t &\geq 0, & \forall t \geq 0 \\ u_t^0 + u_t^1 &= 1, & \forall t \geq 0\,. \end{aligned}$$

(a) Write down the Hamilton-Jacobi-Bellman equation for this problem and derive its first order conditions. .... [2v]

(b) Show that a value function of the from

$$V(t, x) = e^{-\delta t}\left[h(t)\ln(x) + g(t)\right]$$

solves the problem and derive the system of ODEs the deterministic functions h and g must satisfy. .... [2v]

B. Discrete Time Finance (0.5h)

1. Consider a two period binomial model where the initial price of the stock is $S_0 = 100$, $U = 1.5$ and $D = 1.1$. In this market there exists an American put option with a maturity of two periods and exercise price 130. .... [3v]

   (a) Consider that there is a risk-free rate of 20%.

      i. Please find the arbitrage free value (at $t = 0$ and $t = 1$) of the American put option. Describe the optimal exercise policy.

      ii. Compare the American put option value with the one of an European put option with the same maturity and exercise price.

      iii. Is the discounted value of an American put option a Q-martingale? And the stopped process according the the optimal exercise policy?

   (b) Concerning the interest rate, consider that in the first period it is 20% and while in the second period depends on the state of the nature. If the price of the stock goes up the interest rate is 20%; if the price of the stock goes down the interest rate is 30%. Is this market complete? And arbitrage-free?

2. Consider the following problem: $\max_{0 \leq \tau \leq T} E[Z_\tau]$. The optimal value process V is the solution of the following backward recursion

$$
\begin{aligned}
V_n &= \max\{Z_n, E[V_{n+1}|\mathcal{F}_n]\} \\
V_T &= Z_T.
\end{aligned}
$$

   Prove that the optimal value process V is the snell envelope of the payout process Z. .... [2v]

. . . .

Black-Scholes formula:

$$F(t,s) = sN\left[d_1(t,s)\right] - e^{-r(T-t)}KN\left[d_2(t,s)\right].$$

Here N is the cumulative distribution function for the $N[0,1]$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

. . . .